

Schwinger-Dyson Equation, Area Law and Chiral Symmetry in QCD

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Abstract

A Schwinger-Dyson equation for the quark propagator is derived in the context of a Bethe-Salpeter second order formalism developed in preceding papers and of the Minimal Area Law model for the evaluation of the Wilson loop. We discuss how the equal time straight line approximation has to be modified to include correctly trajectories going backwards in time. We also show, by an appropriate selection of the solution of the SD equation, that in the limit of zero quark mass chiral symmetry breaking and a zero mass pseudoscalar meson actually occur. The inclusion of backward quark trajectories proves to be essential to make the model consistent with Goldstone's theorem.

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I. INTRODUCTION

The problem of the chiral symmetry breaking in QCD [1] and that of the consistency with Goldstone's theorem of approximations in the kernels of the Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations has been discussed by various authors [2,3]. To my knowledge, however, in all such papers only the perturbative part of the kernel was actually derived from QCD. To this an *ad hoc* infrared singular term was added only for the sake of convenience and to make the treatment of the resulting equations as easy as possible.

On the other hand, taking advantage of a four-dimensional path integral representation for a *second order* quark-antiquark Green function $H(x_1, x_2; y_1, y_2)$, in a preceding paper we succeeded in obtaining a BS like equation for such quantity entirely from first principles, a part the use of the so called Minimal Area Law (MAL) model for the evaluation of the Wilson loop [4,6].

In this paper I first show that along similar lines a Schwinger-Dyson equation can be derived for the *colorless* second order quark propagator $H(x - y)$ occurring in the BS equation. Then I discuss how the equal time straight line approximation to the minimal area has to be modified in order to make the formalism consistent with Goldstone's theorem in the limit of zero quark masses.

The mentioned path integral representation for $H(x_1, x_2; y_1, y_2)$ is in turn a consequence of the similar Feynmann-Schwinger (FS) representation for a one particle propagator in an external field. Its interest lies in the fact that the gauge field appears in it only in terms of the Wilson loop made by the quark and the antiquark trajectories (connecting y_1 to x_1 and y_2 to x_2) closed by two Schwinger strings [5,6,4],

$$W = \frac{1}{3} \langle \text{TrP} \exp \{ ig \oint_{\Gamma_{\bar{q}q}} dx^\mu A_\mu \} \rangle . \quad (1.1)$$

As usual in (1.1) $A_\mu = \frac{1}{2} A_\mu^a \lambda^a$ is a colour matrix, P the ordering prescription along the loop and the expectation value stands for a functional integration on the gauge field alone. The MAL model consists in assuming that the logarithm of the Wilson loop can be written as the sum of a perturbative and an area term

$$i \ln W = i(\ln W)_{\text{pert}} + \sigma S_{\min} , \quad (1.2)$$

σ being the so called string tension and S_{\min} the minimal area enclosed by $\Gamma_{\bar{q}q}$. In practice S_{\min} is replaced by its equal time straight line approximation, consisting in setting

$$S_{\min} = \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 \delta(z_{10} - z_{20}) |\mathbf{z}_1 - \mathbf{z}_2| \int_0^1 d\lambda \left\{ \dot{z}_{10}^2 \dot{z}_{20}^2 - (\lambda \dot{\vec{z}}_{1T} \dot{\vec{z}}_{20} + (1 - \lambda) \dot{\vec{z}}_{2T} \dot{\vec{z}}_{10})^2 \right\}^{\frac{1}{2}} , \quad (1.3)$$

where $z_1 = z_1(\tau_1)$ and $z_2 = z_2(\tau_2)$ (with $0 \leq \tau_1 \leq s_1$ and $0 \leq \tau_2 \leq s_2$) are the parametric equations for the quark and the antiquark trajectories respectively and $\dot{\vec{z}}_{1T}$, $\dot{\vec{z}}_{2T}$ stand for the transverse components of $\dot{\vec{z}}_1$ and $\dot{\vec{z}}_2$ [$\dot{z}_{jT}^h = (\delta^{hk} - \hat{r}^h \hat{r}^k) \dot{z}_j^h$ with $\vec{r} = \mathbf{z}_1 - \mathbf{z}_2$ and $\hat{r}^h = \frac{r^h}{r}$].

In the limits $x_2 \rightarrow x_1$ and $y_2 \rightarrow y_1$ (i.e. when the initial and respectively the final points of the quark and the antiquark trajectories coincide), Eq. (1.3) is correct only up to the second order terms in \dot{z}_1 and \dot{z}_2 in the general case, but it becomes exact in two different

significant situations: when the two trajectories lie on a plane (with one time and one space dimension) and when they form a double elicoid. This fact may justify its use even in a fully relativistic case.

Notice however that the right hand side of (1.3) is of a purely $q\bar{q}$ interaction type, while the perturbative contribution contains both an $q\bar{q}$ interaction term and two corresponding q and \bar{q} selfinteraction terms. As a consequence of this lack of symmetry, in the limit of zero quark mass (1.3) proves to be inconsistent with the Goldstone theorem, if dynamical symmetry breaking occurs. We shall see that we obtain consistency if we replace (1.3) with

$$\begin{aligned}
S_{\min} = & \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 \delta(z_{10} - z_{20}) |\vec{z}_1 - \vec{z}_2| \epsilon(\dot{z}_{10}) \epsilon(\dot{z}_{20}) \\
& \times \int_0^1 d\lambda \left\{ \dot{z}_{10}^2 \dot{z}_{20}^2 - (\lambda \dot{z}_{1T} \dot{z}_{20} + (1 - \lambda) \dot{z}_{2T} \dot{z}_{10})^2 \right\}^{\frac{1}{2}} - \\
& - \int_0^{s_1} d\tau_1 \int_0^{\tau_1} d\tau'_1 \delta(z_{10} - z'_{10}) |\vec{z}_1 - \vec{z}'_1| \epsilon(\dot{z}_{10}) \epsilon(\dot{z}'_{10}) \\
& \times \int_0^1 d\lambda \left\{ \dot{z}_{10}^2 \dot{z}'_{10}{}^2 - (\lambda \dot{z}_{1T} \dot{z}'_{10} + (1 - \lambda) \dot{z}'_{1T} \dot{z}_{10})^2 \right\}^{\frac{1}{2}} - \\
& - \int_0^{s_2} d\tau_2 \int_0^{\tau_2} d\tau'_2 \delta(z_{20} - z'_{20}) |\vec{z}_2 - \vec{z}'_2| \epsilon(\dot{z}_{20}) \epsilon(\dot{z}'_{20}) \\
& \times \int_0^1 d\lambda \left\{ \dot{z}_{20}^2 \dot{z}'_{20}{}^2 - (\lambda \dot{z}_{2T} \dot{z}'_{20} + (1 - \lambda) \dot{z}'_{2T} \dot{z}_{20})^2 \right\}^{\frac{1}{2}},
\end{aligned} \tag{1.4}$$

where we have used z'_1 and z'_2 for $z_1(\tau'_1)$ and $z_2(\tau'_2)$ respectively and $\epsilon(t)$ for the sign function.

In fact (1.3) breaks down necessarily if the trajectories can go backwards in time. On the contrary, if they never do that, (1.4) becomes identical to (1.3) (since in this case the first term is positive and the other two vanish), but for a plane or an elicoid loop it remains exact, as we shall see, even if they do. The correct inclusion of such more general trajectories turns out to be essential and affects substantially the SD equation.

The resulting SD equation turns out to be equivalent to a system of four non linear integral equations involving four scalar quantities, $h_0(k), \dots, h_3(k)$, which appear as coefficients of product of Dirac matrices with different tensorial character in the expression of the quark propagator. We are not able to handle rigorously such a system. However, its struture suggests the possibility of four different classes of solutions corresponding to different sets of $h_s(k)$ being indentically zero. Actually, if we assume a smooth behaviour as the quark masses go to zero, we can argue that the first class, corresponding to a purely scalar propagator ($h_1 = h_2 = h_3 \equiv 0$), is empty. In fact, if one tries to construct such a solution by iteration, starting from the free propagator, the occurrence in the above limit of the strong infrared singularity due to the area law term (1.4) is found to prevent the procedure to converge.

As an elimination criterion of other classes one can use the unexpected occurrence of zero mass bound states in the corresponding BS equations and the consistency with Goldstone's theorem. Indeed in field theory the occurence of zero mass bound states appears as an exceptional circumstance, expected only in connection with a spontaneously broken symmetry. In this way also the second ($h_3 \equiv 0$) and the third classes ($h_1 = h_2 \equiv 0$) can be rejected as spurious and we are left with the fourth class alone, for which h_0, \dots, h_3 are all different from zero. Inside this fourth class various possible solutions can still be considered, which reduce

to the above classes when the quark masses m_1 and m_2 vanish. By similar arguments also such solutions can be eliminated but one, that for which h_1 and h_2 keep different from zero, while h_3 vanishes for $m_1, m_2 \rightarrow 0$.

The last solution corresponds to a breaking of the chiral symmetry in the vanishing quark mass limit and to the occurrence of a zero mass pseudoscalar solution of the BS equation, as one expects. Notice that what rules out the possibility of a solution of the first class, or reducing to the first class for $m_1, m_2 \rightarrow 0$, is the already mentioned infrared singularity. Therefore this is ultimately responsible for the chiral symmetry breaking.

The plan of the remaining part of the paper is the following: In Sec. II, besides the first and the second order $q\bar{q}$ Green functions $G^{\text{gi}}(x_1, x_2; y_1, y_2)$ and $H^{\text{gi}}(x_1, x_2; y_1, y_2)$, already considered in [4], we introduce the corresponding single quark propagators $G^{\text{gi}}(x - y)$ and $H^{\text{gi}}(x - y)$. In Sec. III we discuss Eq. (1.4); we also introduce the functions $H(x_1, x_2; y_1, y_2)$ and $H(x - y)$, which are obtained by neglecting certain contributions related to the Schwinger strings, but become identical to $H^{\text{gi}}(x_1, x_2; y_1, y_2)$ and $H^{\text{gi}}(x - y)$ in the limits $x_2 \rightarrow x_1$ and $y_2 \rightarrow y_1$ (notice that only such limits are relevant for bound states and chiral symmetry breaking). In Sec. IV we derive a SD equation for $H(x - y)$ along lines similar to those followed to obtain the BS equation for $H(x_1, x_2; y_1, y_2)$ in [4]. In Sec. V we rewrite the BS equation in a form which makes it easier to compare to the SD equation. In Sec. VI we discuss the possible solutions of the SD equation and, in the limit of vanishing quark masses, the chiral symmetry breaking and the BS equation for a zero mass bound state. Finally in Sec. VII we summarize the results and make some additional remarks.

Notice that no attempt of a numerical explicit resolution of the resulting SD equation is made in this paper.

II. GREEN FUNCTIONS AND FEYNMANN-SCHWINGER REPRESENTATIONS

The quark-antiquark and the single quark gauge invariant Green functions are defined as

$$\begin{aligned} G^{\text{gi}}(x_1, x_2, y_1, y_2) &= \frac{1}{3} \langle 0 | T \psi_2^c(x_2) U(x_2, x_1) \psi_1(x_1) \bar{\psi}_1(y_1) U(y_1, y_2) \bar{\psi}_2^c(y_2) | 0 \rangle = \\ &= \frac{1}{3} \text{Tr}_C \langle U(x_2, x_1) S_1(x_1, y_1; A) U(y_1, y_2) \tilde{S}_2(y_2, x_2; -\tilde{A}) \rangle \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} G^{\text{gi}}(x - y) &= \langle 0 | T U(y, x) \psi(x) \bar{\psi}(y) | 0 \rangle = \\ &= i \text{Tr}_C \langle U(y, x) S(x, y; A) \rangle, \end{aligned} \quad (2.2)$$

where ψ^c denotes the charge-conjugate fields, the tilde and Tr_C the transposition and the trace respectively over the color indices alone, U the path-ordered gauge string (Schwinger string)

$$U(b, a) = \text{P exp} \left\{ ig \int_a^b dx^\mu A_\mu(x) \right\} \quad (2.3)$$

(the integration being along the straight line joining a to b , although otherwise specified, and P being the ordering operator on the color matrices), S , S_1 and S_2 the quark propagators in the external gauge field A^μ and obviously

$$\langle f[A] \rangle = \frac{\int \mathcal{D}[A] M_f(A) f[A] e^{iS[A]}}{\int \mathcal{D}[A] M_f(A) e^{iS[A]}}, \quad (2.4)$$

$S[A]$ being the pure gauge field action and $M_f(A)$ the determinant resulting from the explicit integration on the fermionic fields (in practice however $M_f(A) = 1$ in the adopted approximation). Notice that we shall suppress indices specifying the quarks as a rule when dealing with single quark quantities.

The propagator S is supposed to be defined by the equation

$$(i\gamma^\mu D_\mu - m)S(x, y; A) = \delta^4(x - y) \quad (2.5)$$

and the appropriate boundary conditions.

As in [4] we introduce the second order propagator

$$(D_\mu D^\mu + m^2 - \frac{1}{2}g\sigma^{\mu\nu}F_{\mu\nu})\Delta^\sigma(x, y; A) = -\delta^4(x - y) \quad (2.6)$$

and write

$$S(x, y; A) = (i\gamma^\nu D_\nu + m)\Delta^\sigma(x, y; A), \quad (2.7)$$

with $(\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu])$. Then, after replacing (2.7) in (2.2) and (2.1), using an appropriate derivative it is possible to take the differential operator out of the angle brackets and write

$$G^{\text{gi}}(x_1, x_2; y_1, y_2) = -(i\gamma_1^\mu \bar{\partial}_{1\mu} + m_1)(i\gamma_2^\nu \bar{\partial}_{2\nu} + m_2)H^{\text{gi}}(x_1, x_2; y_1, y_2) \quad (2.8)$$

and

$$G^{\text{gi}}(x - y) = (i\gamma_1^\mu \bar{\partial}_\mu + m)H^{\text{gi}}(x - y) \quad (2.9)$$

with

$$H^{\text{gi}}(x_1, x_2; y_1, y_2) = -\frac{1}{3}\text{Tr}_C \langle U(x_2, x_1) \Delta_1^\sigma(x_1, y_1; A) U(y_1, y_2) \tilde{\Delta}_2^\sigma(x_2, y_2; -\tilde{A}) \rangle. \quad (2.10)$$

and

$$H^{\text{gi}}(x - y) = i\text{Tr}_C \langle U(y, x) \Delta^\sigma(x, y; A) \rangle. \quad (2.11)$$

In the last equations $\bar{\partial}_\mu$ stands for the ordinary derivative when acting on the propagators, while it refers to an appropriate distortion of the path when acting on a Schwinger string. In more explicit terms one can also write

$$\bar{\partial}_{b\mu} U(b, a) = \partial_{b\mu} U(b, a) - \int_0^1 d\lambda \lambda (b^\rho - a^\rho) \frac{\delta}{\delta S^{\mu\rho}(a + \lambda(b - a))} U(b, a) \quad (2.12)$$

(with $\delta S^{\mu\nu} = dz^\mu \delta z^\nu - dz^\nu \delta z^\mu$ and the functional derivative being defined through an arbitrary deformation, $z \rightarrow z + \delta z$, of the line connecting a to b starting from the straight line) and a similar expression for $\bar{\partial}_{a^\mu}$.¹

For the second order propagator we have the Feynman-Schwinger representation [4]

$$\Delta^\sigma(x, y; A) = -\frac{i}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp[-i \int_0^s d\tau \frac{1}{2}(m^2 + \dot{z}^2)] \mathcal{S}_0^s \text{P} \exp[ig \int_0^s d\tau \dot{z}^\mu A_\mu(z)], \quad (2.13)$$

with

$$\mathcal{S}_0^s = \text{T} \exp \left[-\frac{1}{4} \int_0^s d\tau \sigma^{\mu\nu} \frac{\delta}{\delta S^{\mu\nu}(z)} \right]. \quad (2.14)$$

T and P being the ordering prescriptions along the path acting on the spin and on the color matrices respectively. Replacing (2.14) in (2.10) and (2.11) we obtain

$$\begin{aligned} H^{\text{gi}}(x_1, x_2; y_1, y_2) &= \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \\ &\quad \exp \left\{ -\frac{i}{2} \int_0^{s_1} d\tau_1 (m_1^2 + \dot{z}_1^2) - \frac{i}{2} \int_0^{s_2} d\tau_2 (m_2^2 + \dot{z}_2^2) \right\} \\ &\quad \times \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \frac{1}{3} \langle \text{Tr P} \exp \{ ig \oint_{\Gamma_{\bar{q}q}} dz^\mu A_\mu(z) \} \rangle, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} H^{\text{gi}}(x - y) &= \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp \left\{ -\frac{i}{2} \int_0^s d\tau (m^2 + \dot{z}^2) \right\} \\ &\quad \times \mathcal{S}_0^s \langle \text{Tr P} \exp \{ ig \oint_{\Gamma_q} dz^\mu A_\mu(z) \} \rangle. \end{aligned} \quad (2.16)$$

Here, the loop $\Gamma_{\bar{q}q}$ occuring in the 4-points function is made by the quark world line γ_1 , the antiquark world line γ_2 followed in the reverse direction, and the two straight lines $x_1 x_2$ and $y_2 y_1$ (Fig. 1), as we already mentioned, on the contrary the loop Γ_q occuring in the 2-points function is made simply by the quark trajectory γ connecting y to x and by the straight line yx (Fig. 2).

¹Given a functional $\Phi[\gamma_{ab}]$ of the curve γ_{ab} with ends a and b , let us assume that the variation of Φ consequent to an infinitesimal modification of the curve $\gamma \rightarrow \gamma + \delta\gamma$ can be expressed as the sum of various terms proportional respectively to δa , to δb and to the single elements $\delta S^{\rho\sigma}(x)$ of the surface swept by the curve. Then, the derivatives $\bar{\partial}_{a^\rho}$, $\bar{\partial}_{b^\rho}$ and $\delta/\delta S^{\rho\sigma}(x)$ are defined by the equation $\delta\Phi = \delta a^\rho \bar{\partial}_{a^\rho} \Phi + \delta b^\rho \bar{\partial}_{b^\rho} \Phi + \frac{1}{2} \int_\gamma \delta S^{\rho\sigma}(x) \delta\Phi/\delta S^{\rho\sigma}(x)$. For a Schwinger string we have $\delta U(b, a) = \delta b^\rho ig A_\rho(b) U(b, a) - \delta a^\rho U(b, a) ig A_\rho(a) + \frac{ig}{2} \int_a^b \delta S^{\rho\sigma}(z) \text{P}(-F_{\rho\sigma}(z) U(b, a))$ and so $\bar{\partial}_{a^\rho} U = -ig U A_\rho(a)$, $\bar{\partial}_{b^\rho} U = ig A_\rho(b) U$ and $\frac{\delta}{\delta S^{\rho\sigma}(z)} U = \text{P}[-ig F_{\rho\sigma}(z) U]$.

III. WILSON LOOP WITH BACKWARD TRAJECTORIES

Now we want to apply Eq. (1.2) to the evaluation of the Wilson loop integral occurring both in (2.15) and in (2.16).

At the lowest order the perturbative term can be written

$$i(\ln W)_{\text{pert}} = -\frac{2}{3}g^2 \oint dz^\mu \oint dz'^\nu D_{\mu\nu}(z - z') \quad (3.1)$$

$D_{\mu\nu}(z - z')$ being the free gauge propagator, which can be grafically represented by a waving line connecting the point z and z' on the loop. If we neglect the contribution coming from lines connecting a point on a trajectory to a point on a string or two point on the strings, we can write for $\Gamma_{\bar{q}q}$

$$\begin{aligned} i(\ln W)_{\text{pert}} = & \frac{4}{3}g^2 \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 D_{\mu\nu}(z_1 - z_2) \dot{z}_1^\mu \dot{z}_2^\nu - \\ & - \frac{4}{3}g^2 \int_0^{s_1} d\tau_1 \int_0^{\tau_1} d\tau'_1 D_{\mu\nu}(z_1 - z'_1) \dot{z}_1^\mu \dot{z}'_1{}^\nu - \frac{4}{3}g^2 \int_0^{s_2} d\tau_2 \int_0^{\tau_2} d\tau'_2 D_{\mu\nu}(z_2 - z'_2) \dot{z}_2^\mu \dot{z}'_2{}^\nu + \dots \end{aligned} \quad (3.2)$$

and for Γ_q

$$i(\ln W)_{\text{pert}} = -\frac{4}{3}g^2 \int_0^s d\tau \int_0^\tau d\tau' D_{\mu\nu}(z - z') \dot{z}^\mu \dot{z}'^\nu + \dots \quad (3.3)$$

Obviously the above approximation would not make sense in general, but we are eventually interested in the two limits $y \rightarrow x$ and $y_2 \rightarrow x_2$, $y_1 \rightarrow x_1$ and in which Eq.'s (3.3) and (3.2) become exact.

Let us pass to consider the area term in (1.2) and refer to the loop $\Gamma_{\bar{q}q}$ first.

In general one can writes

$$S_{\text{min}} = \int_{t_i}^{t_f} dt \int_0^1 d\lambda \left[- \left(\frac{\partial u^\mu}{\partial t} \frac{\partial u_\mu}{\partial t} \right) \left(\frac{\partial u^\nu}{\partial \lambda} \frac{\partial u_\nu}{\partial \lambda} \right) + \left(\frac{\partial u^\mu}{\partial t} \frac{\partial u_\mu}{\partial \lambda} \right)^2 \right]^{\frac{1}{2}}, \quad (3.4)$$

$x^\mu = u^\mu(\lambda, t)$ being the equation of the minimal surface with contour $\Gamma_{\bar{q}q}$. Since (3.4) is invariant under reparametrization, a priori the parameter t can be everything. So we can assume that for fixed t one has

$$u^\mu(1, t) = z_1^\mu(\tau_1(t)), \quad u^\mu(0, t) = z_2^\mu(\tau_2(t)). \quad (3.5)$$

However, if γ_1 and γ_2 never go backwards in time, we can also assume t to be the ordinary time, $u^0(s, t) \equiv t$. Then $\tau_1(t)$ and $\tau_2(t)$ are specified by the equation

$$z_1^0(\tau_1) = z_2^0(\tau_2) = t. \quad (3.6)$$

and the equal time straight line approximation consits in setting

$$u^0(\lambda, t) = t, \quad u^k(\lambda, t) = \lambda z_1^k(\tau_1(t)) + (1 - \lambda) z_2^k(\tau_2(t)). \quad (3.7)$$

Replacing this in (3.4) and performing a change of integration variables, one obtains (1.3).

As regards to Eq. (1.4), notice first that, as we already mentioned, this equation is identical to (1.3), if the trajectories never go backwards in time. In order to discuss the situation we have when this is not the case, we assume that the extreme points coincide ($x_1 = x_2$ and $y_1 = y_2$) and the trajectories lie on a plane. Then let us refer to Fig. 3 and consider first the interaction term. For z_2 between the points R and S we have three different values of τ_1 which solve (3.6) and therefore three different intervals of τ_1 which give contribution; these correspond to z_1 between A and B or between B and C or between C and D . In absolute value such contributions equal respectively the areas RABSR, RCBSR and RCDSR. However, when attention is paid to the sign factors appearing in front of the integral in λ in (1.4), one realizes that the second contribution is negative, while the other two are positive. In this case the algebraic sum of the three areas equals simply the sum of CABC and RCDSR, and this amounts to the total area enclosed in the loop in the strip between the lines RA and SB.

In a similar way, we can see that the contribution to the above strip coming from the self-energy term of particle 1 is also made by the sum of three terms (corresponding to z'_1 between A and B and z_1 between B and C or z'_1 between A and B and z_1 between C and D or z'_1 between B and C and z_1 between C and D), but in this case such sum is zero.

An identical situation occurs for the strip between LE and x F. On the contrary, it is clear that no contribution comes from the interaction term to the area HFGH corresponding to a time larger than $x_1^0 = x_2^0$, while this area is fully taken into account by the self-energy term.

In conclusion, for $\Gamma_{\bar{q}q}$ on a plane and with the extremes of γ_1 and γ_2 coinciding, the Eq. (1.4) is exact. For reason of symmetry the same must be true for a $\Gamma_{\bar{q}q}$ obtained from the preceding one by screwing the trajectories to elics along a line parallel to the time axis. In a general case (1.4) can be only an approximation, but we expect to be a good one, since the contribution coming from trajectories very far from the above classes should be strongly suppressed in the path integral (being far also from the minimum of the exponent).

In a similar way in the case of Γ_q we shall set

$$S_{\min} = - \int_0^s d\tau \int_0^\tau d\tau' \delta(z_0 - z'_0) |\vec{z} - \vec{z}'| \epsilon(\dot{z}_0) \epsilon(\dot{z}'_0) \int_0^1 d\lambda \left\{ \dot{z}_0^2 \dot{z}'_0^2 - \right. \\ \left. - (\lambda \dot{\vec{z}}_T \dot{z}'_0 + (1 - \lambda) \dot{\vec{z}}'_T \dot{z}_0)^2 \right\}^{\frac{1}{2}} \quad (3.8)$$

and notice that even this equation becomes exact for $y = x$ if Γ_q is of one of the types discussed above.

As we shall see, for zero quark masses (1.4) and (3.8), taken together, are approximations consistent with chiral symmetry breaking and Goldstone theorem.

Replacing (3.2) and (1.4) in (2.15) and (3.3) and (3.8) in (2.16) we obtain the following equations

$$H(x_1, x_2; y_1, y_2) = \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int_{y_1}^{x_1} \mathcal{D}z_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \exp \left\{ -\frac{i}{2} \sum_{j=1}^2 \int_0^{s_j} d\tau_j (m_j^2 + \dot{z}_j^2) \right\} \\ \times \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \exp \left\{ i \sum_{j=1}^2 \int_0^{s_j} d\tau_j \int_0^{\tau_j} d\tau'_j E(z_j - z'_j; \dot{z}_j, \dot{z}'_j) - \right. \\ \left. - i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(z_1 - z_2; \dot{z}_1, \dot{z}_2) \right\} \quad (3.9)$$

and

$$H(x-y) = \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \exp \left\{ -\frac{i}{2} \int_0^s d\tau (m^2 + \dot{z}^2) \right\} \\ \times \mathcal{S}_0^s \exp \left\{ i \int_0^s \int_0^\tau E(z-z'; \dot{z}, \dot{z}') \right\}, \quad (3.10)$$

where we have set

$$E(\zeta; p, p') = E_{\text{pert}}(\zeta; p, p') + E_{\text{conf}}(\zeta; p, p') \quad (3.11)$$

with

$$\begin{cases} E_{\text{pert}} &= 4\pi \frac{4}{3} \alpha_s D_{\mu\nu}(\zeta) p^\mu p'^\nu \\ E_{\text{conf}} &= \delta(\zeta_0) |\vec{\zeta}| \epsilon(p_0) \epsilon(p'_0) \int_0^1 d\lambda \{ p_0^2 p'^2 - [\lambda p'_0 \vec{p}_T + (1-\lambda) p_0 \vec{p}_T]^2 \}^{\frac{1}{2}} \end{cases} \quad (3.12)$$

Notice that in principle the quantities $H(x_1, x_2; y_1, y_2)$ and $H(x-y)$ as defined by (3.9)-(3.12) may differ significantly from the original $H^{\text{gi}}(x_1, x_2; y_1, y_2)$ and $H^{\text{gi}}(x-y)$ for arbitrary arguments, since, in writing such equations, we have neglected the contributions to the Wilson loops coming from the strips between x_1^0 and x_2^0 , y_1^0 and y_2^0 , y^0 and x^0 , for what concerns the area parts, and from propagators string-string or string-trajectory for the perturbative parts. However, as we mentioned, the two couples of quantities must coincide in the limits $x_2 \rightarrow x_1$, $y_2 \rightarrow y_1$, and $y \rightarrow x$.

We can also define the quantities

$$G(x_1, x_2; y_1, y_2) = -(i\gamma_1^\mu \partial_{1\mu} + m_1)(i\gamma_2^\nu \partial_{2\nu} + m_2) H(x_1, x_2; y_1, y_2), \quad (3.13)$$

and

$$G(x-y) = i(i\gamma^\mu \partial_\mu + m) H(x-y), \quad (3.14)$$

where we have neglected the integrals along the string occurring in (2.12). Again $G(x_1, x_2; y_1, y_2)$ coincides with $G^{\text{gi}}(x_1, x_2; y_1, y_2)$ and $G(x-y)$ with $G^{\text{gi}}(x-y)$ in the considered limit.

Eq.'s (3.9) and (3.10) are given in terms of purely configurational path integrals. For the need of the following sections path integral representations on the phase space are more convenient. These can be obtained from the preceding ones by performing a Legendre transformation on the exponents (see [4] for details). At the first order in E , or equivalently at the first order in α_s and σa^2 (a being a characteristic length of the problem), we have

$$H(x_1, x_2; y_1, y_2) = \left(\frac{1}{2}\right)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \\ \int_{y_1}^{x_1} \mathcal{D}z_1 \mathcal{D}p_1 \int_{y_2}^{x_2} \mathcal{D}z_2 \mathcal{D}p_2 \exp \left\{ i \sum_{j=1}^2 \int_0^{s_j} d\tau_j [-\dot{z}_j p_j + \frac{1}{2}(p_j^2 - m_j^2)] \right\} \\ \times \mathcal{S}_0^{s_1} \mathcal{S}_0^{s_2} \exp \left\{ i \sum_{j=1}^2 \int_0^{s_j} d\tau_j \int_0^{\tau_j} d\tau'_j E(z_j - z'_j; p_j, p'_j) - \right. \\ \left. - i \int_0^{s_1} d\tau_1 \int_0^{s_2} d\tau_2 E(z_1 - z_2; p_1, p_2) \right\} \quad (3.15)$$

and

$$H(x-y) = \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \mathcal{D}p \exp \left\{ i \int_0^s d\tau [-\dot{z}p + \frac{1}{2}(p^2 - m^2)] \right\} \\ \times \mathcal{S}_0^s \exp \left\{ i \int_0^s d\tau \int_0^\tau d\tau' E(z - z'; p, p') \right\}, \quad (3.16)$$

which is justified by the fact that integration of the momenta in the gaussian approximation reproduces 3.9 and 3.10 up to the mentioned order.

IV. THE SCHWINGER DYSON EQUATION

From (3.16) a Schwinger Dyson equation can be derived by a technique strictly similar to that used in [4] to obtain a Bethe Salpeter equation.

Using a trivial identity we can write with obvious notations

$$H(x-y) = \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \mathcal{D}p \exp \left[i \int_0^s d\tau K \right] \mathcal{S}_0^s \left\{ 1 + \right. \\ \left. + i \int_0^s d\tau \int_0^\tau d\tau' E(z - z'; p, p') \exp \left[i \int_0^\tau d\bar{\tau} \int_0^{\bar{\tau}} d\bar{\tau}' E(\bar{z} - \bar{z}'; \bar{p}, \bar{p}') \right] \right\} = \\ = H_0(x-y) + \frac{i}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \mathcal{D}p \exp \left[i \int_0^s d\tau K \right] \\ \times \int_0^s d\tau \int_0^\tau d\tau' J_{ab}(z - z'; p, p') \sigma^a \mathcal{S}_{\tau'}^\tau \left\{ \exp \left[i \int_{\tau'}^\tau d\bar{\tau} \int_{\tau'}^{\bar{\tau}} d\bar{\tau}' E(\bar{z} - \bar{z}'; \bar{p}, \bar{p}') \right] \right\} \\ \times \sigma^b \mathcal{S}_0^{\tau'} \left\{ \exp \left[i \int_0^{\tau'} d\bar{\tau} \int_0^{\bar{\tau}} d\bar{\tau}' E(\bar{z} - \bar{z}'; \bar{p}, \bar{p}') \right] \right. \\ \left. \times \exp \left[i \int_{\tau'}^\tau d\bar{\tau} \int_0^{\tau'} d\bar{\tau}' E(\bar{z} - \bar{z}'; \bar{p}, \bar{p}') \right] \right\}, \quad (4.1)$$

where

$$K = -p\dot{z} + p^2 - m^2 \quad (4.2)$$

and

$$H_0(x-y) = \frac{1}{2} \int_0^\infty ds \int_y^x \mathcal{D}z \mathcal{D}p \exp \left[i \int_0^s d\tau K \right], \quad (4.3)$$

We have also set $a, b = 0$, $\mu\nu$, with $\sigma^0 = 1$, and have denoted by J_{ab} coefficients which come from E and its commutation with \mathcal{S}_0^s and are given by (see Eq. (A8))

$$J_{0;0} = E(z - z'; p, p') = \\ = 4\pi \frac{4}{3} \alpha_s D_{\alpha\beta}(z - z') p^\alpha p'^\beta + \\ + \sigma \delta(z_0 - z'_0) |\vec{z} - \vec{z}'| \epsilon(p_0) \epsilon(p'_0) \int_0^1 d\lambda \{ p_0^2 p_0'^2 - [\lambda p_0' \vec{p}_T + (1 - \lambda) p_0 \vec{p}_T]^2 \}^{\frac{1}{2}} \\ J_{\mu\nu;0} = -\pi \frac{4}{3} \alpha_s (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) D_{\alpha\beta}(z - z') p'_\beta -$$

$$\begin{aligned}
& -\sigma\delta(z_0 - z'_0)\epsilon(p_0)\frac{(z_\mu - z'_\mu)p_\nu - (z_\nu - z'_\nu)p_\mu}{|\vec{z} - \vec{z}'|\sqrt{p_0^2 - \vec{p}_T^2}}p'_0 \\
J_{0;\rho\sigma} &= \pi\frac{4}{3}\alpha_s p^\alpha(\delta_\rho^\alpha\partial_\sigma - \delta_\sigma^\alpha\partial_\rho)D_{\alpha\beta}(z - z') + \\
& +\sigma\delta(z_0 - z'_0)p_0\frac{(z_\mu - z'_\mu)p'_\nu - (z_\nu - z'_\nu)p'_\mu}{|\vec{z} - \vec{z}'|\sqrt{p_0'^2 - \vec{p}_T'^2}}\epsilon(p'_0) \\
J_{\mu\nu;\rho\sigma} &= -\frac{\pi}{4}\frac{4}{3}\alpha_s(\delta_\mu^\alpha\partial_\nu - \delta_\nu^\alpha\partial_\mu)(\delta_\rho^\alpha\partial_\sigma - \delta_\sigma^\alpha\partial_\rho)D_{\alpha\beta}(z - z')
\end{aligned} \tag{4.4}$$

Up to the first order in E we can replace the last exponential in (4.1) by 1. Then we obtain

$$\begin{aligned}
H(x - y) &= H_0(x - y) + i \int d^4\xi d^4\eta d^4\xi' d^4\eta' H_0(x - \xi) \\
&\times I_{ab}(\xi, \xi'; \eta, \eta') \sigma^a H(\eta - \eta') \sigma^b H(\xi' - y) ,
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
I_{ab}(\xi, \xi'; \eta, \eta') &= \\
&= 4 \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} e^{-ip(\xi-\eta)} J_{ab}\left(\frac{\xi + \eta}{2} - \frac{\xi' + \eta'}{2}; p, p'\right) e^{-ip'(\eta'-\xi')}
\end{aligned} \tag{4.6}$$

and we have used the equation

$$\begin{aligned}
\int_y^x \mathcal{D}z \mathcal{D}p \exp[i \int_0^s d\tau K] \dots &= \int d^4\xi d^4\eta \int \frac{d^4p}{(2\pi)^4} e^{-ip(\xi-\eta)} \\
&\times \int_\xi^x \mathcal{D}z \mathcal{D}p \exp[i \int_\tau^s d\tau' K] \int_y^\eta \mathcal{D}z \mathcal{D}p \exp[i \int_0^\tau d\tau' K] \dots ,
\end{aligned} \tag{4.7}$$

which follows immediately from the discrete form of the functional integral.

Furthermore, by performing a Fourier transform of Eq. (4.5) we obtain

$$\begin{aligned}
\hat{H}(k) &= \hat{H}_0(k) + \\
&+ i \int \frac{d^4l}{(2\pi)^4} \hat{H}_0(k) \hat{I}_{ab}\left(k - l; \frac{k+l}{2}, \frac{k+l}{2}\right) \sigma^a \hat{H}(l) \sigma^b \hat{H}(k).
\end{aligned} \tag{4.8}$$

Finally, by comparison with

$$\hat{H}(k) = \hat{H}_0(k) + i \hat{H}_0(k) \hat{\Gamma}(k) \hat{H}(k) , \tag{4.9}$$

(4.8) gives the Schwinger Dyson equation

$$\hat{\Gamma}(k) = \int \frac{d^4l}{(2\pi)^4} \hat{I}_{ab}\left(k - l; \frac{k+l}{2}, \frac{k+l}{2}\right) \sigma^a \hat{H}(l) \sigma^b . \tag{4.10}$$

In an explicit form the kernels \hat{I}_{ab} are given by

$$\begin{aligned}
\hat{I}_{0;0}(Q; p, p') &= 4 \int d^4\zeta e^{iQ\zeta} E(\zeta; p, p') = 16\pi \frac{4}{3} \alpha_s p^\alpha p'^\beta \hat{D}_{\alpha\beta}(Q) + \\
&\quad + 4\sigma \int d^3\vec{\zeta} e^{-i\vec{Q}\cdot\vec{\zeta}} |\vec{\zeta}| \epsilon(p_0) \epsilon(p'_0) \int_0^1 d\lambda \{p_0^2 p_0'^2 - [\lambda p_0' \vec{p}_T + (1-\lambda) p_0 \vec{p}_T']^2\}^{\frac{1}{2}} \\
\hat{I}_{\mu\nu;0}(Q; p, p') &= 4\pi i \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) p'_\beta \hat{D}_{\alpha\beta}(Q) - \\
&\quad - \sigma \int d^3\vec{\zeta} e^{-i\vec{Q}\cdot\vec{\zeta}} \epsilon(p_0) \frac{\zeta_\mu p_\nu - \zeta_\nu p_\mu}{|\vec{\zeta}| \sqrt{p_0^2 - \vec{p}_T^2}} p'_0 \\
\hat{I}_{0;\rho\sigma}(Q; p, p') &= -4\pi i \frac{4}{3} \alpha_s p^\alpha (\delta_\rho^\beta Q_\sigma - \delta_\sigma^\beta Q_\rho) \hat{D}_{\alpha\beta}(Q) + \\
&\quad + \sigma \int d^3\vec{\zeta} e^{-i\vec{Q}\cdot\vec{\zeta}} p_0 \frac{\zeta_\rho p'_\sigma - \zeta_\sigma p'_\rho}{|\vec{\zeta}| \sqrt{p_0^2 - \vec{p}_T^2}} \epsilon(p'_0) \\
\hat{I}_{\mu\nu;\rho\sigma}(Q; p, p') &= \pi \frac{4}{3} \alpha_s (\delta_\mu^\alpha Q_\nu - \delta_\nu^\alpha Q_\mu) (\delta_\rho^\beta Q_\sigma - \delta_\sigma^\beta Q_\rho) \hat{D}_{\alpha\beta}(Q)
\end{aligned} \tag{4.11}$$

where in the second and in the third equation $\zeta_0 = 0$ has to be understood.

Notice that, were 4.10 covariant, i.e. were the quantities \hat{I}_{ab} tensors, we could write

$$i\hat{H}^{-1}(k) = a(k^2) + kb(k^2). \tag{4.12}$$

However, due to the equal time straight line approximation, that is not the case; we must assume (4.11) to hold in the meson center of mass frame and set

$$i\hat{H}^{-1}(k) = \sum_{r=0}^3 \omega_r(k) h_r(k), \tag{4.13}$$

with

$$\omega_0 = 1, \quad \omega_1 = \gamma^0, \quad \omega_2 = -\vec{\gamma} \cdot \hat{k}, \quad \omega_3 = \gamma^0 \vec{\gamma} \cdot \hat{k}, \tag{4.14}$$

$\hat{k} = \frac{1}{|\vec{k}|} \vec{k}$ and $h_0(k), \dots, h_3(k)$ functions of k_0 and $|\vec{k}|$ separately. Obviously (4.13) reduces to (4.12) for $h_0 = a(k^2)$, $h_1 = k_0 b(k^2)$, $h_2 = |\vec{k}| b(k^2)$, $h_3 = 0$.

From (4.9) and (4.13) it follows

$$\hat{\Gamma}(k) = i\hat{H}^{-1}(k) - i\hat{H}_0^{-1}(k) = h_0(k) - (k^2 - m^2) + \sum_{r=1}^3 \omega_r(k) h_r(k) \tag{4.15}$$

and

$$\hat{H}(k) = i \frac{h_0(k) - \sum_{r=1}^3 \omega_r(k) h_r(k)}{h_0^2(k) - h_1^2(k) + h_2^2(k) - h_3^2(k)}. \tag{4.16}$$

Replacing such equations in (4.10) and taking advantage of

$$\frac{1}{4} \text{Tr}(\omega_r^+ \omega_s) = \delta_{rs}, \tag{4.17}$$

we finally obtain

$$h_r(k) = \delta_{r0}(k^2 - m^2) - i \sum_{s=0}^3 \int \frac{d^4 l}{(2\pi)^4} \frac{R_{rs}(k, l) h_s(l)}{h_0^2(l) - h_1^2(l) + h_2^2(l) - h_3^2(l)}, \quad (4.18)$$

with

$$R_{rs}(k, l) = \mp \frac{1}{4} \hat{I}_{ab}(k - l; \frac{k+l}{2}, \frac{k+l}{2}) \text{Tr}[\omega_r^+(k) \sigma^a \omega_s(k) \sigma^b], \quad (4.19)$$

where the sign - applies to the $s = 0$ case, the sign + to all the other cases. Notice that from well known properties of Dirac's matrices it follows immediately $R_{01} = R_{10} = R_{02} = R_{20} = R_{13} = R_{31} = R_{23} = R_{32} = 0$, while from

$$\hat{I}_{\mu\nu;0}(Q; q, q) = -\hat{I}_{0;\mu\nu}(Q; q, q) \quad \text{and} \quad \hat{I}_{\mu\nu;\rho\sigma}(Q; q, q) = \hat{I}_{\rho\sigma;\mu\nu}(Q; q, q) \quad (4.20)$$

it follows also $R_{03} = R_{30} = 0$. Actually only R_{00} , R_{11} , R_{12} , R_{21} , R_{22} , R_{33} are different from zero. Expressions for these quantities in terms of the \hat{I}_{ab} are given in Appendix B. Here, as an example, we want only to report the explicit expression

$$R_{00}(k, l) = -4\pi \frac{4}{3} \alpha_s [4 \frac{p^2 l^2 - (pl)^2}{(p-l)^4} + \frac{3}{4}] - \sigma \int d^3 \vec{\zeta} e^{-i(\vec{k}-\vec{l}) \cdot \vec{\zeta}} |\vec{\zeta}| |(k_0 + l_0)| \sqrt{(k_0 + l_0)^2 - (\vec{k}_T + \vec{l}_T)^2}. \quad (4.21)$$

The confining term in this equation contains an infrared singularity essentially of the type

$$\frac{\partial^2}{\partial \epsilon^2} \frac{1}{(\vec{k} - \vec{l})^2 + \epsilon^2}, \quad (4.22)$$

with $\epsilon \rightarrow 0$. Similar singularities appear in the other diagonal kernels R_{rr} .

Expressions like (4.22) correspond to well defined distributions and are harmless if they appear in integrals multiplied by regular functions. However, they can give troubles and generate new singularities when occurring in connection with discontinuous functions. As it appears from (4.21) and (4.11), R_{rs} have also a bad ultraviolet behaviour that should be handled by renormalization. Renormalization would affect the perturbative parts of R_{rs} , but its explicit consideration is not important for our present purpose. So we shall simply suppose to regulate (4.18) by a cut-off Λ .

Finally going back to the first order propagator (3.14) we can also set

$$iG^{-1}(k) = \sum_{r=0}^3 \omega_r(k) g_r(k), \quad (4.23)$$

and then we have

$$\begin{cases} h_0 = mg_0 + k_0 g_1 + |\vec{k}| g_2 \\ h_1 = mg_1 + k_0 g_0 - |\vec{k}| g_3 \\ h_2 = mg_2 - |\vec{k}| g_0 + k^0 g_3 \\ h_3 = mg_3 + |\vec{k}| g_1 + k_0 g_2 \end{cases} \quad (4.24)$$

From (4.23) it is apparent that, for zero quark masses, chiral symmetry breaking corresponds to $g_0(k) \neq 0$ and/or $g_3(k) \neq 0$ and consequently from (4.24) to $h_1(k)$ and $h_2(k)$ not simultaneously vanishing.

V. THE BETHE-SALPETER EQUATION

The Bethe-Salpeter equation for the quantity $H(x_1, x_2; y_1, y_2)$ is derived from (3.9) by applying the same identity used in (4.1) to the interaction, rather than to the selfenergy term, as done in the case of the Schwinger-Dyson equation. In terms of the quantities defined by Eq.'s (4.6) and (4.4) it reads

$$\begin{aligned} H(x_1, x_2; y_1, y_2) &= H_1(x_1 - y_1) H_2(x_2 - y_2) - \\ &-i \int d^4\xi_1 d^4\xi_2 d^4\eta_1 d^4\eta_2 H_1(x_1 - \xi_1) H_2(x_2 - \xi_2) \\ &\times I_{ab}(\xi_1, \xi_2; \eta_1, \eta_2) \sigma_1^a \sigma_2^b H(\eta_1, \eta_2; y_1, y_2), \end{aligned} \quad (5.1)$$

where H_1 and H_2 denote the quark and the antiquark propagators as defined by (3.10).

The corresponding homogenous equation in the momentum representation is

$$\begin{aligned} \Phi_P(k) &= -i \int \frac{d^4u}{(2\pi)^4} \hat{H}_1\left(\frac{1}{2}P + k\right) \hat{H}_2\left(\frac{1}{2}P - k\right) \\ &\hat{I}_{ab}\left(k - u, \frac{1}{2}P + \frac{k+u}{2}, \frac{1}{2}P - \frac{k+u}{2}\right) \sigma_1^a \sigma_2^b \Phi_P(u), \end{aligned} \quad (5.2)$$

the center of mass frame being understood, i.e. $P = (m_B, \vec{0})$.

The wave function $\Phi_P(k)$ can be also reinterpreted as a matrix (in which the column index refers to the quark and the row one to the antiquark respectively) and the pedices 1 and 2 in the spin operators suppressed. Then setting

$$\Phi_P(k) = \hat{H}_1\left(\frac{1}{2}P + k\right) \Gamma_P^M(k) \hat{H}_2\left(-\frac{1}{2}P + k\right) C \quad (5.3)$$

and using $C \hat{H}^T(k) C^{-1} = \hat{H}^T(-k)$ we can rewrite (5.2) as

$$\begin{aligned} \Gamma_P^M(k) &= -i \int \frac{d^4u}{(2\pi)^4} \hat{I}_{ab}\left(k - u, \frac{1}{2}P + \frac{k+u}{2}, \frac{1}{2}P - \frac{k+u}{2}\right) \\ &\sigma^a \hat{H}_1\left(\frac{1}{2}P + u\right) \Gamma_P^M(u) \hat{H}_2\left(-\frac{1}{2}P + u\right) \bar{\sigma}^b \end{aligned} \quad (5.4)$$

with

$$\bar{\sigma}^0 = \sigma^0, \quad \bar{\sigma}^{\mu\nu} = C(\sigma^{\mu\nu})^T C^{-1} = -\sigma^{\mu\nu}, \quad (5.5)$$

C being the charge conjugation matrix. This is the form of the BS-equation more convenient for our present purposes.

More specifically we need to consider the case $P = 0$, corresponding to a zero mass bound state. If we take into account the property (cf. (4.11))

$$\hat{I}_{ab}(Q; p, p') \bar{\sigma}^b = -\hat{I}_{ab}(Q; p, -p') \sigma^b, \quad (5.6)$$

and assume $m_1 = m_2 = m$, we have

$$\Gamma_0^M(k) = i \int \frac{d^4u}{(2\pi)^4} \hat{I}_{ab}\left(k - u, \frac{k+u}{2}, \frac{k+u}{2}\right) \sigma^a \hat{H}(u) \Gamma_0^M(u) \hat{H}(u) \sigma^b. \quad (5.7)$$

Notice that a zero bound state exists, if (5.7) has a non trivial solution.

VI. CHIRAL SYMMETRY BREAKING AN GOLDSTONE THEOREM

To discuss Eq.'s (4.10) or(4.18) and (5.7) it is convenient to rewrite them in euclidean form. This can be obtained by making the substitutions $p_0 \rightarrow ip_4$ in all momentum variables and by setting $\gamma_0 = i\gamma_4$. To solve the ambiguities related to the occurrence sign funtions and the square roots in (4.11) one should go back to the eucleadean counterparts of the original equations and particularly of (3.15) and (3.16). Alternatively one can extract from the square roots factors as $|p_0|$ and then set $\epsilon(p_0)|p_0| = p_0$ before performing the substitution. It also convenient to redefine h_1 as ih_1 , h_3 as ih_3 and also R_{12} as iR_{12} and R_{21} as $-iR_{21}$. The resulting expression for R_{00} is, e.g.,

$$R_{00}(k, l) = -4\pi\frac{4}{3}\alpha_s[4\frac{p^2l^2-(pl)^2}{(p-l)^4} + \frac{3}{4}] + \sigma \int d^3\vec{\zeta} e^{-i(\vec{k}-\vec{l})\cdot\vec{\zeta}} |\vec{\zeta}| |(k_4 + l_4)| \sqrt{(k_4 + l_4)^2 + (\vec{k}_T + \vec{l}_T)^2} . \quad (6.1)$$

while expressions for the confinement parts of the other R_{rs} are reported in appendix C.

Then, making explicit the terms that are different from zero, (4.18) can be rewritten

$$\begin{cases} h_0(k) = -k^2 - m^2 + \int \frac{d^4l}{(2\pi)^4} \frac{R_{00}(k,l) h_0(l)}{h_0^2(l) + h_1^2(l) + h_2^2(l) + h_3^2(l)} \\ h_1(k) = \int \frac{d^4l}{(2\pi)^4} \frac{R_{11}(k,l) h_1(l) + R_{12}(k,l) h_2(l)}{h_0^2(l) + h_1^2(l) + h_2^2(l) + h_3^2(l)} \\ h_2(k) = \int \frac{d^4l}{(2\pi)^4} \frac{R_{21}(k,l) h_1(l) + R_{22}(k,l) h_2(l)}{h_0^2(l) + h_1^2(l) + h_2^2(l) + h_3^2(l)} \\ h_3(k) = \int \frac{d^4l}{(2\pi)^4} \frac{R_{33}(k,l) h_3(l)}{h_0^2(l) + h_1^2(l) + h_2^2(l) + h_3^2(l)} \end{cases} \quad (6.2)$$

Similarly for the euclidean counterpart of (5.7) we have

$$\Gamma_0^M(k) = \int \frac{d^4u}{(2\pi)^4} \hat{I}_{ab}(k - u, \frac{k+u}{2}, \frac{k+u}{2}) \sigma^a \frac{h_0(u) + \gamma_4 h_1(u) + \vec{\gamma} \cdot \hat{u} h_2(u) + \gamma_4 \vec{\gamma} \cdot \hat{u} h_3(u)}{h_0^2(u) + h_1^2(u) + h_2^2(u) + h_3^2(u)} \\ \Gamma_0^M(u) \frac{h_0(u) + \gamma_4 h_1(u) + \vec{\gamma} \cdot \hat{u} h_2(u) + \gamma_4 \vec{\gamma} \cdot \hat{u} h_3(u)}{h_0^2(u) + h_1^2(u) + h_2^2(u) + h_3^2(u)} \sigma^b . \quad (6.3)$$

Notice that *a priori* Eq. (6.2) could be expected to admit various types of solutions corresponding to different sets of h_s identically vanishing

$$\begin{aligned} \text{A1)} & \quad h_1 = h_2 = h_3 = 0, \\ \text{A2)} & \quad h_1, h_2 \neq 0, \quad h_3 = 0, \\ \text{A3)} & \quad h_1 = h_2 = 0, \quad h_3 \neq 0, \\ \text{A4)} & \quad h_1, h_2, h_3 \neq 0. \end{aligned} \quad (6.4)$$

To gain more insight on the nature of such solutions let us assume that we can solve (6.2) by iteration starting by an initial ansatz of the type

$$h_0^{(0)} = -k^2 - m^2, \quad h_r^{(0)} = \mu_r^2 f_r\left(\frac{k}{\bar{\mu}}\right), \quad (r = 1, 2, 3) \quad (6.5)$$

μ_1, μ_2, μ_3 and $\bar{\mu}$ being constants with the dimensions of a mass and f_r arbitrary functions chosen in such a way that at least the first iteration is meaningfull. E. g. after the first iteration we have for h_0

$$h_0^{(1)}(k) = -k^2 - m^2 - \int \frac{d^4 l}{(2\pi)^4} \frac{R_{00}(k, l) (l^2 + m^2)}{(l^2 + m^2)^2 + [h_1^{(0)}(l)]^2 + [h_2^{(0)}(l)]^2 + [h_3^{(0)}(l)]^2}. \quad (6.6)$$

Solutions of the types A1-A4) can be obtained from (6.5) simply by taking $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_1, \mu_2 \neq 0$, $\mu_3 = 0$ etc. respectively. Notice that $\mu_1, \dots, \bar{\mu}$, when different from zero must be expressed in terms of the masses existing in the theory, m , $\sqrt{\sigma}$ and the cut-off Λ or the renormalization scale.

Let us consider separately the various cases.

Case A1). In this case we expect no actual solution smooth in $m = 0$. In fact, setting $m = 0$ and $h_1 = h_2 = h_3 = 0$ Eq. (6.6) becomes

$$h_0^{(1)}(k) = -k_4^2 - \vec{k}^2 - \int \frac{d^4 l}{(2\pi)^4} \frac{R_{00}(k, l)}{l_4^2 + \vec{l}^2}. \quad (6.7)$$

Having in mind Eq. (4.21) we realize that integration over l_4 in such an equation produces a $\frac{1}{|l|}$ singularity in the tridimensional integral which conspires with (4.22) for $\vec{k} \rightarrow 0$ bearing a $\frac{1}{k^2}$ singularity in $h_0^{(1)}(k)$. Similar circumstances occur at various steps in the iteration rendering the entire process unstable and apparently preventing convergence.

Case A2). One can immediately check that, if h_3 vanishes, setting

$$\Gamma_0^M(k) = [\gamma_4 h_1(k) + \vec{\gamma} \cdot \hat{k} h_2(k)] \gamma_5, \quad (6.8)$$

(6.3) is made identical to the subsystem formed by the second and the third Eq. (6.2) and so non trivially verified. Therefore, a zero mass pseudoscalar meson exists, independently of the quark mass value and of the occurrence of a specific symmetry. This class has to be rejected.

Case A3). This time a non trivial solution of (6.3) can be obtained by setting

$$\Gamma_0^M(k) = \gamma_4 h_3(k). \quad (6.9)$$

This would correspond to a zero mass scalar meson. Also this class must be considered spurious.

We are left with case A4 in which all the quantities h_1 , h_2 and h_3 are different from zero. Inside this class we can still distinguish various, possibly single, solutions corresponding to different behaviours of the h_s in the limit $m \rightarrow 0$. Schematically we can think of generating such different solutions by setting in (6.5)

$$\begin{aligned} \text{B1)} & \quad \mu_1 = \mu_2 = \mu_3 = m, \\ \text{B2)} & \quad \mu_1 = \mu_2 = \sqrt{\sigma}, \quad \mu_3 = m, \\ \text{B3)} & \quad \mu_1 = \mu_2 = m, \quad \mu_3 = \sqrt{\sigma}, \\ \text{B4)} & \quad \mu_1 = \mu_2 = \mu_3 = \sqrt{\sigma}, \end{aligned} \quad (6.10)$$

Notice that B1, B2, B3 correspond to solutions that are reduced respectively to the types A1, A2, A3 for $m \rightarrow 0$; in fact the factors m in front of the various $f_r(k)$ are reproduced at every step of the iteration. Notice also that for $m \rightarrow 0$ the theory acquires chiral symmetry. As we mentioned the symmetry is broken if in the same limit h_1 and h_2 do not simultaneously

vanish. So we have a broken chiral symmetry in cases B2 and B4, actual symmetry in the other two cases.

Let us consider again the various cases.

Ansatzes B1 and B3 reproduce similar situations as A1 and A3 for $m \rightarrow 0$ and can be excluded for the same reasons. Ansatz B4 contradicts Goldstone's theorem; in fact for $m \rightarrow 0$, h_3 keeps different from zero in this case and the expression (6.8) would not be a solution of (6.3). On the contrary ansatz B2 seems to satisfy all requirements. For $m \rightarrow 0$ (6.8) becomes a non trivial solution of (6.3) and a zero mass pseudo scalar bound state occurs.

In conclusion we are left with solution B2 alone. This seems to be the only one which is physically sensible and mathematically consistent. For light u and d quarks it would correspond to a breaking of the approximate chiral symmetry and would correctly provide a π meson with a small mass.

VII. CONCLUSIONS

Using the formalism developped in [4], the minimal area law and the straight line approximation in a modified form, we have obtained the Schwinger-Dyson equation (4.10) for the quark propagator occurring in the Bethe-Salpeter equation (5.1) or (5.4). We have explicitly rewritten the SD equation as a system of non linear integral equations involving four scalar functions. However, we found that such system has not a unique solution and had to resort to additional criteria to select the correct one.

After a first selection, assuming an iterative resolution procedure to converge, we have been brought to consider various possible solutions corresponding to different starting ansatzes. One of these (solution B2) produces chiral symmetry breaking in the quark mass vanishing limit and a zero mass pseudoscalar meson. This corresponds to what we believe to be the real situation. All the other solutions are mathematically inconsistent, or in contradiction with general theorems, or, simply, have unexpected features which we do not believe should be present in real QCD. If we accept B2 to be the correct solution, then, we achieve a very consistent formalism for the treatment of the quark-antiquark bound states in QCD, both for light and for heavy quarks.

We stress once more that the occurrence in the kernels of the infrared singularity (4.22) is what seems to prevent solutions A1 or B1 from existing. Therefore a strict connection appears to exist between confinement as expressed by the area law and chiral symmetry breaking.

Notice, also, that the possibility of a solution with the properties we have found depends strictly on the assumption of (1.4) in place of (1.3). Had we used straight line approximation in the original form, only the perturbative parts of (4.11) would have occurred in (4.10). Then the kernels in (5.7) would have been different from the kernels in (4.11) and (6.8) would not have been a solution of (5.7). The correct solution of (6.2) would have been of the type A1 and chiral symmetry breaking and the zero mass pseudo scalar meson would not have occurred.

Let us make a final comment.

As we mentioned the confinement parts of the kernels \hat{I}_{ab} , as given by (4.11), are not tensors. As we mentioned, this circumstance is a consequence of the equal time straight line

approximation, which is obviously frame dependent. We have assumed the privileged frame to be the center of mass frame of the meson. But then \hat{I}_{ab} can be rewritten as tensors, if we introduce explicitly the total four-momentum of the meson P . For this aim it is sufficient to set

$$Q_{\parallel}^{\rho} = \frac{Q \cdot P}{P^2} P^{\rho}, \quad Q_{\perp}^{\rho} = Q^{\rho} - Q_{\parallel}^{\rho}, \quad (7.1)$$

give analogous definitions for p and p' and make the identifications

$$Q_0^2 \rightarrow Q_{\parallel}^2, \quad \vec{Q}^2 \rightarrow -Q_{\perp}^2 \quad \text{etc.} \quad (7.2)$$

and similar ones.

In the above perspective the quark propagator $\hat{H}(k)$ should be rewritten as $\hat{H}(P, k)$ and the Eq. (4.13) as

$$i\hat{H}^{-1}(P, k) = \sum_{r=0}^3 \omega'_r(P, k) h'_r(P, k), \quad (7.3)$$

with

$$\omega'_0 = 1, \quad \omega'_1 = \frac{1}{\sqrt{P^2}} \not{P}, \quad \omega'_2 = \frac{1}{\sqrt{k^2}} \not{k}, \quad \omega'_3 = \frac{-i}{2\sqrt{P^2 k^2}} [\not{P}, \not{k}] \quad (7.4)$$

and (6.2) correspondingly modified.

Since the quark propagator has been originally defined in an independent way by (2.11), at first sight it can be surprising that now it is made dependent on the meson momentum. Notice, however, that it acquires again P dependence as a consequence of the straight line approximation. After that it becomes the specific propagator to be used in that BS equation.

Notice also that the need to choose solution B2 (with $h_3 \neq 0$ for $m \neq 0$) is a consequence of the above dependence. Had \hat{I}_{ab} 's been tensors in k and l alone (as in the abelian case), we should have had $h_3 = 0$ (see Eq. (4.12)) and A1 would have been the only possible sensible solution. Therefore a P dependence of the quark propagator seems to be essential for a consistent chiral symmetry breaking and not an artifact of the minimal area law model or of the straight line approximation.

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APPENDIX A: FUNCTIONAL DERIVATIVES

Taking into account appendix C of [4], and specifically Eq. (C10), we can write for the perturbative part

$$\begin{aligned}
\frac{\delta}{\delta S^{\mu\nu}(z(\tau))} \int_a^b d\bar{\tau} \int_{a'}^{b'} d\bar{\tau}' E_{\text{pert}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') &= \\
&= 4\pi \frac{4}{3} \alpha_s \frac{\delta}{\delta S^{\mu\nu}(z(\tau))} \int_a^b d\bar{\tau} \int_{a'}^{b'} d\bar{\tau}' D_{\alpha\beta}(\bar{z} - \bar{z}') \dot{\bar{z}}^\alpha \dot{\bar{z}}'^\beta = \\
&= 4\pi \frac{4}{3} \alpha_s \left\{ \chi_{(a,b)}(\tau) \int_{a'}^{b'} d\bar{\tau}' (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) D_{\alpha\beta}(\bar{z} - \bar{z}') \dot{\bar{z}}'^\beta - \right. \\
&\quad \left. - \chi_{(a',b')}(\tau) \int_a^b d\bar{\tau} (\delta_\mu^\beta \partial_\nu - \delta_\nu^\beta \partial_\mu) D_{\alpha\beta}(\bar{z} - \bar{z}') \dot{\bar{z}}^\alpha \right\}, \tag{A1}
\end{aligned}$$

where $\chi_{(a,b)}(\tau)$ and $\chi_{(a',b')}(\tau)$ are the characteristic functions of the intervals (a, b) and (a', b') ((a, b) and (a', b') contained in $(0, s)$) and the derivatives are intended to act on the entire argument of $D_{\alpha\beta}$. Due to the arbitrariness of the above intervals, we can also write formally

$$\begin{aligned}
\frac{\delta}{\delta S^{\mu\nu}(z(\tau))} E_{\text{pert}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') &= \\
&= 4\pi \frac{4}{3} \alpha_s [\delta(\tau - \bar{\tau}) (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) D_{\alpha\beta}(\bar{z} - \bar{z}') \dot{\bar{z}}'^\beta - \\
&\quad - \delta(\tau - \bar{\tau}') (\delta_\mu^\beta \partial_\nu - \delta_\nu^\beta \partial_\mu) D_{\alpha\beta}(\bar{z} - \bar{z}') \dot{\bar{z}}^\alpha]. \tag{A2}
\end{aligned}$$

For the confining part for consistency we must define

$$\frac{\delta}{\delta S^{\mu\nu}(z(\tau))} \int_a^b d\bar{\tau} \int_{a'}^{b'} d\bar{\tau}' E_{\text{conf}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') \tag{A3}$$

as the the straight line approximation of $\frac{\delta}{\delta S^{\mu\nu}} S_{aa'}^{bb'}$, the quantity $S_{aa'}^{bb'}$ denoting the area of the strip of the minimal area enclosed by Γ_q delimited by the two geodetics connecting $z(a)$ to $z(a')$ and $z(b)$ to $z(b')$ respectively. Then, having again in mind Eq. (C9) of [4], we have

$$\begin{aligned}
\frac{\delta}{\delta S^{\mu\nu}(z(\tau))} \int_a^b d\bar{\tau} \int_{a'}^{b'} d\bar{\tau}' E_{\text{conf}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') &= \\
&= \sigma \chi_{(a,b)}(\tau) \chi_{(a',b')}(\tau') \epsilon(\dot{z}_0) \epsilon(\dot{z}'_0) \left(\frac{(z^\mu - z'^\mu) \dot{z}^\nu - (z^\nu - z'^\nu) \dot{z}^\mu}{[-\dot{z}^2 (z - z')^2 + (\dot{z}(z - z'))^2]^{\frac{1}{2}}} - \right. \\
&\quad \left. - \frac{(z^\mu - z'^\mu) \dot{z}'^\nu - (z'^\nu - z'^\nu) \dot{z}'^\mu}{[-\dot{z}'^2 (z - z')^2 + (\dot{z}'(z - z'))^2]^{\frac{1}{2}}} \right) \Big|_{z_0(\tau')=z_0(\tau)} = \\
&= \sigma \left\{ \chi_{(a,b)}(\tau) \int_{a'}^{b'} d\bar{\tau}' \delta(z_0 - \bar{z}'_0) |\dot{z}'_0| \epsilon(\dot{z}_0) \epsilon(\dot{z}'_0) \frac{(z^\mu - \bar{z}'^\mu) \dot{z}^\nu - (z^\nu - \bar{z}'^\nu) \dot{z}^\mu}{[\dot{z}^2 (\bar{z} - \bar{z}')^2 + (\dot{z}(\bar{z} - \bar{z}'))^2]^{\frac{1}{2}}} - \right. \\
&\quad \left. - \chi_{(a',b')}(\tau) \int_a^b d\bar{\tau} \delta(\bar{z}_0 - z_0) |\dot{z}_0| \epsilon(\dot{z}_0) \epsilon(\dot{z}_0) \frac{(\bar{z}^\mu - z^\mu) \dot{z}^\nu - (\bar{z}^\nu - z^\nu) \dot{z}^\mu}{[\dot{z}^2 (\bar{z} - z)^2 + (\dot{z}(\bar{z} - z))^2]^{\frac{1}{2}}} \right\} \tag{A4}
\end{aligned}$$

or

$$\begin{aligned}
\frac{\delta}{\delta S^{\mu\nu}(z(\tau))} E_{\text{conf}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') &= \\
&= \sigma \delta(\tau - \bar{\tau}) \delta(z_0 - \bar{z}'_0) \epsilon(\dot{z}_0) \dot{z}'_0 \frac{(\bar{z}^\mu - \bar{z}'^\mu) \dot{z}^\nu - (\bar{z}^\nu - \bar{z}'^\nu) \dot{z}^\mu}{|\bar{z} - \bar{z}'| \sqrt{\dot{z}_0^2 - \dot{z}_T^2}} - \\
&\quad - \sigma \delta(\tau - \bar{\tau}') \delta(\bar{z}_0 - z_0) \epsilon(\dot{z}'_0) \dot{z}_0 \frac{(\bar{z}^\mu - \bar{z}'^\mu) \dot{z}'^\nu - (\bar{z}^\nu - \bar{z}'^\nu) \dot{z}'^\mu}{|\bar{z} - \bar{z}'| \sqrt{\dot{z}_0'^2 - \dot{z}_T'^2}} \tag{A5}
\end{aligned}$$

Notice also

$$\begin{aligned}
\frac{\delta^2}{\delta S^{\rho\sigma}(z(\tau')) \delta S^{\mu\nu}(z(\tau))} E_{\text{pert}}(\bar{z} - \bar{z}', \dot{\bar{z}}, \dot{\bar{z}}') &= \\
&= -8\pi \frac{4}{3} \alpha_s [\delta(\tau - \bar{\tau}) \delta(\tau' - \bar{\tau}') (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) (\delta_\rho^\beta \partial_\sigma - \delta_\sigma^\beta \partial_\rho) D_{\alpha\beta}(\bar{z} - \bar{z}')] \tag{A6}
\end{aligned}$$

and

$$\frac{\delta^2}{\delta S^{\rho\sigma}(z(\tau'))\delta S^{\mu\nu}(z(\tau))}E_{\text{conf}}(\bar{z} - \bar{z}', \dot{z}, \dot{z}') = 0. \quad (\text{A7})$$

Then

$$\begin{aligned} l\mathcal{S}_0^s E(z - z', \dot{z}, \dot{z}')(\mathcal{S}_0^s)^{-1} &= E - \frac{1}{4}\sigma^{\mu\nu} \int_0^s d\theta \left[\frac{\delta}{\delta S^{\mu\nu}(\zeta(\theta))}, E \right] + \\ &+ \frac{1}{32}\sigma^{\mu\nu}\sigma^{\rho\sigma} \int_0^s d\theta \int_0^\theta d\theta' \left[\frac{\delta}{\delta S^{\rho\sigma}(\zeta(\theta'))}, \left[\frac{\delta}{\delta S^{\mu\nu}(\zeta(\theta))}, E \right] \right] + \dots = \\ &= E(z - z', \dot{z}, \dot{z}') - \frac{1}{4}\sigma^{\mu\nu} \left\{ 4\pi \frac{4}{3}\alpha_s (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) D_{\alpha\beta}(z - z') \dot{z}'^\beta + \right. \\ &\quad + \sigma \delta(z_0 - z'_0) \epsilon(\dot{z}_0) \dot{z}'_0 \frac{(z^\mu - z'^\mu) \dot{z}^\nu - (z^\nu - z'^\nu) \dot{z}^\mu}{|\vec{z} - \vec{z}'| \sqrt{\dot{z}_0^2 - \dot{z}'_T^2}} - \\ &\quad - 4\pi \frac{4}{3}\alpha_s (\delta_\mu^\beta \partial_\nu - \delta_\nu^\beta \partial_\mu) D_{\alpha\beta}(z - z') \dot{z}^\alpha - \\ &\quad \left. - \sigma \delta(z_0 - z'_0) \epsilon(\dot{z}'_0) \dot{z}_0 \frac{(z^\mu - z'^\mu) \dot{z}'^\nu - (z^\nu - z'^\nu) \dot{z}'^\mu}{|\vec{z} - \vec{z}'| \sqrt{\dot{z}_0^2 - \dot{z}'_T^2}} \right\} - \\ &- \frac{1}{16} 4\pi \frac{4}{3} \alpha_s \sigma^{\mu\nu} \sigma^{\rho\sigma} (\delta_\mu^\alpha \partial_\nu - \delta_\nu^\alpha \partial_\mu) (\delta_\rho^\beta \partial_\sigma - \delta_\sigma^\beta \partial_\rho) D_{\alpha\beta}(z - z') \end{aligned} \quad (\text{A8})$$

Finally, after the Legendre transformation, \dot{z}^μ and \dot{z}'_μ are simply replaced in (A8) by p^μ and p'^μ .

APPENDIX B: EXPLICIT EXPRESSION OF THE SD KERNELS

Evaluating explicitly the trace in Eq. (4.19), one obtains

$$\begin{cases} R_{00}(k, l) = -\hat{I}_{0;0} - 2g^{\mu\rho} g^{\nu\sigma} \hat{I}_{\mu\nu;\rho\sigma} \\ R_{11}(k, l) = \hat{I}_{0;0} - 8g^{\nu\sigma} \hat{I}_{0\nu;0\sigma} + 2g^{\mu\rho} g^{\nu\sigma} \hat{I}_{\mu\nu;\rho\sigma} \\ R_{12}(k, l) = -R_{21}(l, k) = 4i\hat{I}_{0;0i} \hat{l}^i - 8g^{\nu\sigma} \hat{I}_{0\nu;i\sigma} \hat{l}^i \\ R_{22}(k, l) = \hat{I}_{0;0} \hat{k} \cdot \hat{l} + 8g^{\nu\sigma} \hat{I}_{i\nu;j\sigma} \hat{k}^i \hat{l}^j + 2g^{\mu\rho} g^{\nu\sigma} \hat{I}_{\mu\nu;\rho\sigma} \hat{k} \cdot \hat{l} + 2i(\hat{I}_{0;ij} - \hat{I}_{ij;0}) \hat{k}^i \hat{l}^j \\ R_{33}(k, l) = \hat{I}_{0;0} - 8(g^{ij} + 2\hat{k}^i \hat{l}^j) \hat{I}_{0i;0j} + 4g^{\nu\sigma} \hat{I}_{i\nu;j\sigma} \hat{k}^i \hat{l}^j + 2g^{\mu\rho} g^{\nu\sigma} \hat{I}_{\mu\nu;\rho\sigma} \end{cases} \quad (\text{B1})$$

where the arguments are intended as in (4.18).

APPENDIX C: EUCLIDEAN FORM OF THE CONFINING KERNELS

The euclidean form of $\hat{R}_{rs}(k, l)$ is obtained from Eq. (B1) and (4.11) by setting $p = p' = q$ and then making the substitutions $Q_0 \rightarrow iQ_4$, $q_0 \rightarrow iq_4$, $R_{12} \rightarrow iR_{12}$, $R_{21} \rightarrow -iR_{21}$, according to the rules given in Sec. IV.

We report the explicit expression for the confinement parts, the only non trivial:

$$\left\{ \begin{array}{l}
R_{00}^{\text{conf}}(k, l) = \sigma \int d^3 \vec{\zeta} e^{-i \vec{Q} \cdot \vec{\zeta}} |\vec{\zeta}| |q_4| \sqrt{q_4^2 + \vec{q}_T^2} \\
R_{11}^{\text{conf}}(k, l) = -\sigma \int d^3 \vec{\zeta} e^{-i \vec{Q} \cdot \vec{\zeta}} |\vec{\zeta}| |q_4| \sqrt{q_4^2 + \vec{q}_T^2} \\
R_{12}^{\text{conf}}(k, l) = R_{21}^{\text{conf}}(l, k) = -4i\sigma \int d^3 \vec{\zeta} e^{-i \vec{Q} \cdot \vec{\zeta}} \epsilon(q_4) \frac{q_4^2}{|\vec{\zeta}|} \frac{\vec{\zeta} \cdot \vec{l}}{\sqrt{q_4^2 + \vec{q}_T^2}} \\
R_{22}^{\text{conf}}(k, l) = -\sigma \int d^3 \vec{\zeta} e^{-i \vec{Q} \cdot \vec{\zeta}} |q_4| [|\vec{\zeta}| \sqrt{q_4^2 + \vec{q}_T^2} + 4i \frac{(\vec{\zeta} \cdot \vec{k})(\vec{q} \cdot \vec{l}) - (\vec{\zeta} \cdot \vec{l})(\vec{q} \cdot \vec{k})}{|\vec{\zeta}| \sqrt{q_4^2 + \vec{q}_T^2}}] \\
R_{33}^{\text{conf}}(k, l) = -\sigma \int d^3 \vec{\zeta} e^{-i \vec{Q} \cdot \vec{\zeta}} |\vec{\zeta}| |q_4| \sqrt{q_4^2 + \vec{q}_T^2}
\end{array} \right. \quad (\text{C1})$$

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FIGURES

FIG. 1. Wilson loop $\Gamma_{q\bar{q}}$ for the four points Green function formed by the quark and the antiquark trajectories (full lines) closed by two Swinger strings (broken lines).

FIG. 2. Wilson loop Γ_q for the quark propagator formed by the quark trajectory closed by a Swinger string.

FIG. 3. Loop $\Gamma_{q\bar{q}}$ for coinciding end points with the quark trajectory going also backwards in time.